

Trace of BV -functions on irregular subsets

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1 Introduction.

The purpose of the paper is to generalize all the main results on boundary trace of the book [5], Chapter 6, to a wider class of sets. This chapter is an extended version of the earlier publication [3]. Our paper is an extended and completed version of our publication [2], where some results were presented without proofs or in a weakened form. In [3], [5], boundary trace was defined for regions Ω with finite perimeter (in the sense of Caccioppoli–De Giorgi) and the main results about trace were obtained under an additional assumption that normals in the sense of Federer exist almost everywhere on $\partial\Omega$. Instead of that, here we suppose that $\partial\Omega$ is a countably $(n-1)$ -rectifiable set, which is a more general condition. Readers can get acquainted with the theory of sets of finite perimeter and BV functions in the books [5], [4], [10].

The analytical tools we use are basically the same as in [3], [5]. Relations between isoperimetric inequalities and integral inequalities (of Sobolev embedding theorems type) play an essential role. First these relations were discovered by V. Maz'ya [6]. Almost all results formulated below are valid not only for regions in \mathbb{R}^n but for regions on C^1 -smooth n -dimensional manifolds as well. This becomes clear from Corollary 2.

In fact, deep knowledge in geometric measure theory, in particular, in rectifiable currents is not necessary. All necessary (very restricted) information from this theory are given below.

Let us explain the reason for our results to generalize those in [3]. It is known that the boundary ∂E of a set $E \subset \mathbb{R}^n$ with a finite perimeter consists of two parts. One of them, so called reduced boundary ∂^*E , consists of all points at which normals in the sense of Federer exist. It is known that this part is a countably $(n-1)$ -rectifiable set. The perimeter $P(E)$ of a set E equals $H_{n-1}(\partial^*E)$, where H_k is k -dimensional Hausdorff measure. So the requirement, that the normals in the sense of Federer exist a.e. on $\partial^*\Omega$ is equivalent to the condition that $\partial\Omega = \partial^*\Omega$. For sure, all sets are considered up to sets of $(n-1)$ -dimensional Hausdorff measure zero.

In general, $\partial E \setminus \partial^*E$ consist of two parts, a countably $(n-1)$ -rectifiable set and so called completely unrectified (irregular) set $\text{Ir}(E)$. The latter may

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have either finite, or infinite $(n - 1)$ -dimensional Hausdorff measure. The assumption that $\partial\Omega$ is a countably $(n - 1)$ -rectifiable set means that the set Ir is empty. However even in this case the countably rectifiable set $\partial\Omega$ can be essentially vaster than ∂^*E .

Let us explain this situation by the following example. Consider an open disk in a plane with a sequence of intervals I_i removed. Suppose that the union of these intervals is closed. The results of [3] on boundary traces are not applicable to such a region Ω (the intervals do not belong to the reduced boundary) but the boundary of Ω is a countably 1-rectifiable set.

Note by the way that even for a smooth function on Ω its limits at the points of the intervals I_i from right and left can be different, so that it is reasonable to introduce traces with two different values in some points.

Notations. Denote by $A\Delta B$ the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of A and B . H_k denotes the k -dimensional Hausdorff measure and $\text{Vol}(A)$ denotes the Lebesgue measure of $A \subset \mathbb{R}^n$ or, equivalently, its n -dimension Hausdorff measure.

The dimension $k = n - 1$ will play a special role for us and to be short we denote $H_{n-1} = \mu$. From here on words “almost all”, “measurable”, etc, will be used with respect either to H_n , or to $H_{n-1} = \mu$, it will be clear from the context to which one.

Denote by $B_p(r)$ the open ball of radius r centered at p and by $\bar{B}_p(r)$ its closure.

$\Theta_A(p, k)$ denotes density with respect the measure H_k of a set A at p ; i.e.,

$$\Theta_A(p, k) = \lim_{r \rightarrow 0} v_k^{-1} r^{-k} H_k(A \cap B_p(r)),$$

where v_k is the volume of the unit ball in \mathbb{R}^k . Note, that we use basically not densities, but one-sided densities in the paper, see the next section.

Countably rectifiable sets. There are several equivalent definitions of countably (k, H_k) -rectifiable sets. One can find a detailed exposition in H. Federer’s monograph [8], Chapter 3, and more specifically 3.2.19, 3.2.25, 3.2.29.

The following definition is the most convenient for our purposes

Definition 1. The measurable set $A \subset \mathbb{R}^n$ is called countably (k, H_k) -rectifiable if there exists a sequence of C^1 -smooth k -dimensional surfaces M_i , $i = 1, 2, \dots$, such that A can be decomposed $A = \bigcup_{i=0}^{\infty} A_i$, where $\mu(A_0) = 0$ and $A_i \subset M_i$ for $i > 0$. Moreover, the sets A_i can be chosen such that the following conditions hold:

$$\Theta_A(p, k) = 0, \quad \Theta_{(A \setminus A_i)}(p, k) = 1 \tag{1}$$

for almost all $p \in A_i$.

We need the case $k = n - 1$ only, so we call countably $(n - 1, \mu)$ -rectifiable set *countably rectifiable* to be short.

Any countably rectifiable set A has almost everywhere so called the approximative tangent $(n - 1)$ -plane $T_p A$, which coincides with the tangent plane to M_i at p . A point at which $T_p A$ exists and, in addition, equality (1) holds is called the regular point. Thus, almost all (by measure μ) points of A are regular. We drop a definition of $T_p A$ because we need only the following its property: for every sequence of positive numbers $r_j \rightarrow 0$, there exist positive numbers $\epsilon_j \rightarrow 0$ such that

$$\lim_{r_j \rightarrow 0} r^{1-n} \mu(B_p(r_j) \setminus L_{r_j \epsilon_j}) = 0, \quad (2)$$

where L_δ is the δ -neighborhood of $T_p A$. If ν is a normal to $T_p A$ at p we will say that ν is a normal to A at p .

Functions. As usually, $BV(\Omega)$ means the class of locally summable in Ω functions such that their gradients are vector charges. Denote by $\chi(E)$ the characteristic function of E and by $P_\Omega(E)$ the perimeter of $E \subset \Omega$; i.e., $P_\Omega(E) = \|\chi_E\|_{BV(\Omega)}$. (We use notation $\|f\|_{BV(\Omega)} = \text{var grad } f(\Omega)$.) For more details see [5], [3], [10], [4].

We will need the Fleming–Rishel formula [9]

$$\|f\|_{BV(\Omega)} = \int_{-\infty}^{\infty} P_\Omega(E_t) dt, \quad (3)$$

where $f \in BV(\Omega)$, $E_t = \{x \mid f(x) > t\}$, and also the following formula closely connected with it

$$\nabla f(E) = \int_{-\infty}^{+\infty} \nabla \chi_{E_t}(E) dt, \quad (4)$$

where E is ant measurable subset of Ω , see for instance Theorem 14 in [3] or Lemma 6.6.5/1 in [5].

Remark 1. We will often consider sets E for which $P_\Omega(E) < \infty$. For instance, it can be sets E_t of points where a function f greater than t . If considerations are local then the finiteness perimeter condition can be replaced by the assumption that a set $E \cap \Omega$ has *locally finite perimeter*; i.e., $P_{\Omega \cap Q}(E) < \infty$ for any bounded region Q .

2 One-sided densities

Let us consider a measurable set $E \subset \mathbb{R}^n$. Let ν be a unit vector at a point $x \in \mathbb{R}^n$. Denote $B_x^\nu(r) = B_x(r) \cap \{y \mid (y - x)\nu \geq 0\}$. The limit

$$\Theta_E^\nu(x) = \lim_{r \rightarrow 0} 2v_n^{-1} r^{-n} H_n(B_x^\nu(r) \cap E).$$

is called *one-sided density* of the set E at x with respect to ν .

Upper and lower one-sided densities $\overline{\Theta}_E^\nu(x)$, $\underline{\Theta}_E^\nu(x)$ are defined analogically as upper and lower limits. Now let x be a regular point of the countably rectifiable set A . Then there are two normals to A at x and, correspondingly, it is naturally to consider two one-sided densities with respect to A , namely $\Theta_E^\nu(x)$ and $\Theta_E^{-\nu}(x)$.

We often consider the boundary of Ω in the capacity of A assuming that the boundary is a countably rectifiable set. In such cases we suppose usually that $E \subset \Omega$.

Remark 2. It is easy to see that if a set G is measurable and $\Theta_G^\nu(x) = 1$ then

$$\Theta_E^\nu(x) = \lim_{r \rightarrow 0} \frac{H_n(B_x^\nu(r) \cap G \cap E)}{H_n(B_x^\nu(r) \cap G)} = \lim_{r \rightarrow 0} 2v_n^{-1} r^{-n} H_n(B_x^\nu(r) \cap G \cap E). \quad (5)$$

The following statement is a simple corollary of the isoperimetric inequality for subsets of a ball.

Lemma 1. *Let E be an measurable set with a finite perimeter, $Q = \{x \in \mathbb{R}^n \mid \sum x_i^2 < 1, a < x_n < 1\}$, where $a \leq 1/2$. Then the following isoperimetric inequality holds*

$$\min\{H_n(Q \cap E), H_n(Q \setminus E)\} \leq c_n P_Q(E)^{\frac{n}{n-1}}, \quad (6)$$

where $c_n > 0$ depends on dimension only.

Lemma 2. *Let the boundary of a region Ω is a countably rectifiable set. Then either $\Theta_\Omega^\nu(x) = 1$, or $\Theta_{\mathbb{R}^n \setminus \Omega}^\nu(x) = 1$ at each regular point $x \in \partial\Omega$ and for every normal $\nu(x)$ to $\partial\Omega$.*

Note that for normals ν , $-\nu$, any combination of values 0 and 1 for one-sided densities are possible. That can happen even on a set of positive μ -measure.

Proof. Let ν be a normal at a regular point $x \in \partial\Omega$. Consider semi-balls $B_i^\nu = B_x^\nu(r_i)$, where $r_i \rightarrow +0$ as $i \rightarrow \infty$. Denote by C_i intersection of $\epsilon_i r_i$ -neighborhood of the plane T_x with B_i^ν , $A_i = B_i^\nu \setminus C_i$. It is clear that $\text{Vol}(C_i) < v_{n-1} \epsilon_i r_i^n$. By (2) the inequalities $P_{A_i} \leq \mu(A_i \cap \partial\Omega) < \epsilon r_i^{n-1}$ hold for large i and sufficiently small ϵ_i . Now the lemma follows immediately from the isoperimetric inequality (6) applied to the region A_i and the set $A_i \cap \Omega$. \square

Example 1. Consider a sequence of small bubbles (disjoint round balls) $B_{x_i}(r_i)$ located in the unit open ball $B_0(1)$. It is easy to choose these bubbles in such a way that all the points $p \in S_0(1)$ are the limits of some subsequences of the bubbles and, besides, there is no other limit points. In addition, suppose that the radii of these balls vanish so fast that $\sum_i r_i^{n-1} < \infty$.

Define $\Omega = \bigcup B_{x_i}(r_i)$. Its boundary is rectifiable. This set is not connected but in dimensions $n > 2$, one can connect the bubbles by very thin tubules such that the new set Ω (completed with bubbles) becomes a region with rectifiable boundary. The sphere $S_0(1)$ belongs to the boundary of Ω . So almost all the points of this sphere are regular points of $\partial\Omega$. However they do not belong to the reduced boundary of Ω ; i.e., the set $S_0(1) \cap \partial^*\Omega$ is empty. Moreover, bubbles can be chosen in such a way that at every point x of the sphere $S_0(1)$, the condition $\Theta_\Omega^\nu(x) = 0$ holds for every normal.

Denote by Γ the set of all points $x \in \partial\Omega$ such that $\Theta_\Omega^\nu(x) = 1$ for at least one normal ν . It is not difficult to see that $\partial^*\Omega \subset \Gamma$. Indeed, the vector ν_F is the normal in the sense of Federer if and only if $\Theta_\Omega^{-\nu_F}(x) = 1$ and $\Theta_\Omega^{\nu_F}(x) = 0$.

Remark 3. It is well known that $P(\Omega) = \mu(\partial^*\Omega)$. Recall that if $P(\Omega) < \infty$, then $\text{var} \nabla \chi_\Omega(\partial\Omega \setminus \partial^*\Omega) = 0$ and

$$\nabla \chi_\Omega(E) = - \int_E \nu_F(x) \mu(dx) \quad (7)$$

for any measurable set $E \subset \partial^*\Omega$, see for instance [3], Theorem 6.2.2/1.

Lemma 3. *Any countably rectifiable set A can be equipped with measurable field ν of (unit) normals.*

Proof. The set A , up to a subset of measure 0, is located on $(n-1)$ -dimensional C^1 -smooth manifolds M_i of some countable family. It is not difficult to see that almost each point $x \in A$ belongs to only one surface M_i . Let us orient every manifold M_i by a continuous field of normals. Since the approximative tangent plane to A at x coincides with the tangent plane $T_x M_i$ and the intersection $A \cap M_i$ is measurable, we obtain a measurable field of normals to A by choosing normals $\nu(x)$ to M_i in the capacity of normals to A . \square

Remark 4. It is clear that a measurable vector field of unit normals is not unique, there are infinitely many of such vector fields. Let us fix some vector field ν constructed in Lemma 3. It is not only measurable but is located on C^1 -smooth surfaces M_i from a chosen family and continuous along every such surface. Besides, if a countably rectifiable set A is the boundary of a region Ω , $A = \partial\Omega$, then the vector field ν can be chosen so that, at points $x \in \partial^*\Omega$, vectors $\nu(x)$ is directed opposite to normals in the sense of Federer. A vector field having such properties is called *standard*.

Lemma 4. *Let A be a countably rectifiable set, ν be a measurable field of normals to A , and E be a measurable subset of \mathbb{R}^n . Then the sets $\{x \in A \mid \Theta_E^\nu(x) = 1\}$ and $\{x \in A \mid \Theta_E^\nu(x) = 0\}$ are measurable.*

Proof. First assume that vector field ν is standard and a family of surfaces $\{M_i\}$ is chosen as above, in Remark 4. The sets $M_i \cap A$ are measurable. The functions $\phi_i^r(x) = 2v_n^{-1}r^{-n}H_n(B_x^\nu(r) \cap E)$ defined on $M_i \cap A$ are continuous. In particular they are measurable. Let us extend these functions to all A by zero. Their sum $\phi^r = \sum_i \phi_i^r$ defined on A is measurable too. Therefore, the functions $\underline{\phi}(x) = \liminf_{r \rightarrow 0} \phi^r(x)$ and $\overline{\phi}(x) = \limsup_{r \rightarrow 0} \phi^r(x)$ are measurable and hence the sets

$$\{x \in A \mid \Theta_E^\nu(x) = 0\} = \{x \in A \mid \underline{\phi}(x) = 0\},$$

$$\{x \in A \mid \Theta_E^\nu(x) = 1\} = \{x \in A \mid \overline{\phi}(x) = 1\}$$

are measurable. The same holds for the field $-\nu$ as well. Now let $\tilde{\nu}$ be any measurable unit vector field of normals to $\partial\Omega$. Then the sets $\{\nu = \tilde{\nu}\}$ and $\{-\nu = \tilde{\nu}\}$ are measurable, and thereby the set $\{x \in A \mid \Theta_E^{\tilde{\nu}}(x) = 0\}$ and $\{x \in A \mid \Theta_E^{\tilde{\nu}}(x) = 1\}$ are measurable too. \square

Let a set A be countably rectifiable, $P(E) < \infty$, and ν be a normal to A at x . Denote

$$\begin{aligned} \partial_A^\nu E &= \{x \in A \mid \Theta_E^\nu(x) = 1\}, \\ \partial_A^1 E &= (\partial_A^\nu E) \cup (\partial_A^{-\nu} E), \quad \partial_A^2 E = (\partial_A^\nu E) \cap (\partial_A^{-\nu} E). \end{aligned} \quad (8)$$

Roughly speaking, $\partial_A^1 E$ is the set of points of A such that E “adjoins” to A with one-sided density 1 at least from one side and $\partial_A^2 E$ is the part of A such that E “adjoins” with one-sided density 1 from both sides.

Note that the following formulas hold:

$$\Gamma = \partial_{\partial\Omega}^1 \Omega, \quad \partial_{\partial\Omega}^\nu E = \partial_\Gamma^\nu E. \quad (9)$$

We will use Lemma 6.6.3/1 from [5] (or, that is the same, Lemma 13 from [3]). The lemma is about the trace of a characteristic function. As the notion of trace be introduced later, we formulate the lemma in a convenient form.

Lemma 5. *Let $P(\Omega) < \infty$, $E \subset \Omega$, $P_\Omega(E) < \infty$. Then for almost all $x \in \partial^*\Omega$*

$$\chi_{\partial^*E}(x) = \lim_{r \rightarrow 0} \frac{\int_{B_x(r)} \chi_E dx}{\text{Vol}(B_x(r) \cap \Omega)} = \lim_{r \rightarrow 0} \frac{\text{Vol}(B_x(r) \cap E)}{\text{Vol}(B_x(r) \cap \Omega)}. \quad (10)$$

For sure, only the first equality is essential, while the latter one is trivial.

Remark 5. In Lemma 5, the condition $E \subset \Omega$ can be dropped if one replaces E to $E \cap \Omega$ and the condition $P_\Omega(E) < \infty$ to $P(E) < \infty$.

The following lemma is the key one for our subsequent considerations.

Lemma 6. *Let A be a countably rectifiable set, ν be a measurable field of normals along A , and $P(E) < \infty$. Then μ -almost everywhere on A , one-sided densities $\Theta_E^\nu(x)$ equal either 0, or 1.*

Proof. It suffices to prove the lemma for standard normal vector fields and taking into account only regular points of A (see Lemma 3 and Remark 4).

1. First let A be C^1 -smooth $(n-1)$ -dimensional manifold M . Since our statement is local, we can suppose that M divides some its neighborhood bounded by a smooth hypersurface onto two semi-neighborhoods, Ω_1 and Ω_2 . Set $E_i = \Omega_i \cap E$, $i = 1, 2$. It is clear that $P(E_i) < \infty$.

Note that $\chi_{\partial^* E_1}(x)$ equals 1 if $x \in \partial^* E_1 \cap M$ and equals 0 if $x \in M \setminus \partial^* E_1$. Therefore, applying Lemma 5 to the sets $E = E_1$ and $\Omega = \Omega_1$ and Remark 2 for $G = \Omega_1$, we see that for almost all points $x \in M$ the one-sided density $\Theta_{E_1}^\nu(x)$ is equal either 0 or 1, where ν is the normal M directed to the side of Ω_1 . The same is true for E_2 and Ω_2 . Finally, since

$$1 \geq \Theta_E^\nu(x) = \Theta_{E_1}^\nu(x) + \Theta_{E_2}^\nu(x),$$

we see that the lemma is proved for $A = M$.

2. Let us pass to the general case. Let $\{M_i\}$ be a family of C^1 -smooth submanifolds, mentioned in the definition of standard normal fields. In the item 1, the lemma was already proved for each M_i . The intersection $A \cap M_i$ is μ -measurable, and one-sided density at a point depends on ν and E only. Thus $\Theta_E^\nu(x)$ is equal either 0 or 1 almost everywhere on $A \cap M_i$. Since A , up to a set of measure 0, coincides with the union of sets $A \cap M_i$, the lemma is proved. \square

Corollary 1. *Let Ω be a region such that its boundary is a countably rectifiable set. If $E \subset \Omega$ and $P(E) < \infty$, then for any (measurable) field ν of normals to $\partial\Omega$, one-sided densities Θ_E^ν are equal almost everywhere either 0 or 1.*

It is clear now, that, for the reduced boundary of any set E with $P(E) < \infty$, the following holds:

$$A \cap \partial^* E = (\partial_A^1 E) \setminus (\partial_A^2 E), \quad (11)$$

in particular

$$\partial^* \Omega = (\partial_\Gamma^1 \Omega) \setminus (\partial_\Gamma^2 \Omega). \quad (12)$$

Corollary 2. *Let x be a regular point of $\partial\Omega$. Suppose that $\Theta_{G_1}^\nu(x) = \Theta_{G_2}^\nu(x) = 1$ for some sets G_1, G_2 . In addition, assume that there is a family of sets $\mathcal{B}_x^\nu(r)$ such that*

$$B_x(\rho_1(r)) \cap G_1 \subset \mathcal{B}_x^\nu(r) \subset B_x(\rho_2(r)) \cap G_2, \quad (13)$$

where $\rho_2(r) \rightarrow 0$ as $r \rightarrow 0$. Then the equality

$$\Theta_E^\nu(x) = \lim_{r \rightarrow 0} \frac{H_n(\mathcal{B}_x^\nu(r) \cap E)}{H_n(\mathcal{B}_x^\nu(r))} \quad (14)$$

holds for any set $E \subset \mathbb{R}^n$ with finite perimeter.

This corollary allows to consider one-sided densities for sets with finite perimeters in any C^1 -smooth manifold with a continuous metric tensor. Therefore further considerations are applicable not only to \mathbb{R}^n , but also to any such a manifold.

3 Trace on a countably rectifiable set

Here we define trace on a countably rectifiable set for a function defined in Ω . Within this section we do not require function to belong to $BV(\Omega)$. Instead of that we only suppose that the sets $E_t = \{x \in \Omega \mid f(x) > t\}$ have finite perimeters for almost all t . We call functions *BV-similar* if they have such property. (As it was mentioned in Remark 1, it would be sufficiently to suppose that E_t has locally finite perimeter.)

Let a countably rectifiable set A is contained in the closure $\bar{\Omega}$ of a region Ω . Let us define trace¹ $f^\nu(x)$ with respect to normal ν at $x \in \partial_A^\nu \Omega$ for a *BV-similar* function f as follows:

$$f^\nu(x) = \sup\{t \mid x \in \partial_A^\nu E_t\}.$$

We can suppose (this change nothing), that supremum is taken only over t such that $P(E_t) < \infty$. Moreover we assume that $\sup \emptyset = -\infty$.

Let us emphasize, that trace is defined not everywhere on A . However if one extend f to all \mathbb{R}^n (for instance, by a constant), so that $A = \partial_A^\nu(\mathbb{R}^n \setminus A)$, then f^ν is defined on A everywhere.

In the case $x \in \partial_A^2 \Omega$ we also define the upper and lower traces by equations

$$f^*(x) = \max\{f^\nu(x), f^{-\nu}(x)\}, \quad f_*(x) = \min\{f^\nu(x), f^{-\nu}(x)\}.$$

If $x \in A \cap \partial^* \Omega = (\partial_A^1 \Omega) \setminus (\partial_A^2 \Omega)$, we put $f^*(x) = f^\nu(x)$, where $-\nu$ is the normal in the sense of Federer. In this case we do not define $f_*(x)$ at all. However, if f is extended on all \mathbb{R}^n (for instance, by a constant) then $A = \partial_A^2(\mathbb{R}^n \setminus A) = \partial_A^1(\mathbb{R}^n \setminus A)$ and the upper and lower traces are defined on all A .

It is clear, that $f^*(x) = \sup\{t \mid x \in \partial_1^1 E_t\}$, $f_*(x) = \sup\{t \mid x \in \partial_1^2 E_t\}$.

¹Our terminology is different of one in [3], [5]. Namely, we use terms trace and average trace instead of rough trace and trace.

Lemma 7. *Let $A \subset \bar{\Omega}$ be a countably rectifiable set, ν be a measurable field of normals to A . Then for any BV -similar function f its trace f^ν on $\partial_A^\nu \Omega$ is measurable and*

$$\mu(\{x \in \partial_A^\nu \Omega \mid f^\nu(x) \geq t\}) = \mu(\partial_A^\nu E_t) \quad (15)$$

for almost all $t \in \mathbb{R}$.

Remark 6. 1) Analogously to Lemma 7, it can be proved that traces f^* and f_* are measurable as well and

$$\mu(\{x \in \partial_\Gamma^1 \Omega \mid f^*(x) \geq t\}) = \mu(\partial_\Gamma^1 E_t), \quad (16)$$

$$\mu(\{x \in \partial_\Gamma^2 \Omega \mid f_*(x) \geq t\}) = \mu(\partial_\Gamma^2 E_t). \quad (17)$$

2) In fact, instead of (15), we will prove, that

$$\mu\left(\{x \in \partial_A^\nu \Omega \mid f^\nu(x) \geq t\} \Delta \partial_A^\nu E_t\right) = 0$$

for all t except a countable subset.

3) Note that in (15)–(17) unstrict inequalities can be replaced by strict ones.

Proof. Denote $B_t = \{x \in \partial_A^\nu \Omega \mid f^\nu(x) \geq t\}$, $Y_t = \partial_A^\nu E_t$ and $X_t = B_t \setminus Y_t$. It is easy to see that $B_t \supset Y_t$. Thus, it remains to prove that $\mu(X_t) = 0$.

The sets Y_t are measurable, and the sets X_t are disjoint. It is not difficult to see that the inclusions $Y_{t_0} \supset Y_{t_1}$ and $Y_{t_0} \cup X_{t_0} \supset Y_{t_1} \cup X_{t_1}$ hold for $t_0 < t_1$. The latter inclusion implies that $Y_{t_0} \supset X_{t_1}$. So

$$\left(\bigcap_{t < t_1} Y_t\right) \setminus Y_{t_1} \supset X_{t_1}.$$

From the other hand the sets $(\bigcap_{t < t_1} Y_t) \setminus Y_{t_1}$ are measurable and disjoint. Therefore $\mu((\bigcap_{t < t_1} Y_t) \setminus Y_{t_1}) = 0$ for almost all $t_1 \in \mathbb{R}$. From this it follows that the sets X_t are subsets of measure zero sets for almost all $t \in \mathbb{R}$. In particular, they are measurable. It follows that the sets B_t are measurable. \square

Lemma 8. *Let $A \subset \bar{\Omega}$ be a countably rectifiable set, f be a BV -similar function. Then the inequality*

$$-f^\nu(x) = (-f)^\nu(x) \quad (18)$$

holds for almost all $x \in \partial_A^\nu \Omega$.

Proof. Lemma 8 is equivalent to the statement that the equality

$$\sup\{t \mid x \in \partial_A^\nu E_t\} = \inf\{t \mid x \in \partial_A^\nu (\Omega \setminus E_t)\}.$$

holds for almost all $x \in A$. The last equality means that

$$\sup\{t \mid \underline{\Theta}_{E_t}^\nu(x) = 1\} = \inf\{t \mid \underline{\Theta}_{(\Omega \setminus E_t)}^\nu(x) = 1\}.$$

In its turn, this is equivalent to the equality

$$\sup\{t \mid \underline{\Theta}_{E_t}^\nu(x) = 1\} = \inf\{t \mid \overline{\Theta}_{E_t}^\nu(x) = 0\}.$$

Denote by L and R the left and the right parts of the last equality. It is not difficult to see that the functions $\overline{\Theta}_{E_t}^\nu(x)$ and $\underline{\Theta}_{E_t}^\nu(x)$ are not increasing in t . Therefore $L \leq R$. Consider the set of the points x such that $L(x) < R(x)$. It suffices to prove that μ -measure of this set equals zero.

For this let us choose a countable everywhere dense set $\{t_i\}_{i=1}^\infty$ such that $P(E_{t_i}) < \infty$. If $L(x) < R(x)$ then there exists t_i such that $L(x) < t_i < R(x)$. Now our assertion follows from Lemma 6 applied to the set E_{t_i} . \square

Corollary 3. *For any BV-similar function f and for almost all $x \in A$ the following equalities hold:*

$$(f^\nu)^+ = (f^+)^\nu, \quad (f^\nu)^- = (f^-)^\nu. \quad (19)$$

Proof. The first equality can be derived directly from definitions. The latter one easily follows from Lemma 8. Indeed,

$$(f^-)^\nu = ((-f)^+)^\nu = ((-f)^\nu)^+ = (-(f^\nu))^+ = (f^\nu)^-.$$

\square

Lemma 9. *For any BV-similar functions f, g and almost all $x \in A$ the following equality holds:*

$$(f + g)^\nu(x) = f^\nu(x) + g^\nu(x). \quad (20)$$

Proof. First prove that $(f + g)^\nu(x) \geq f^\nu(x) + g^\nu(x)$ for all $x \in \Gamma$. Indeed, choose numbers $F < f^\nu(x)$ and $G < g^\nu(x)$ such that the sets $E_F^f = \{x \mid f(x) > F\}$ and $E_G^g = \{x \mid g(x) > G\}$ have finite perimeters. Then $\Theta_{E_F^f}^\nu(x) = 1$ and $\Theta_{E_G^g}^\nu(x) = 1$.

Denote $W = E_{F+G}^{f+g}$. We have

$$W = \{x \mid f(x) + g(x) > F + G\} \supset E_F^f \cap E_G^g.$$

Therefore $\Theta_W^\nu(x) = 1$ and so

$$(f + g)^\nu(x) = \sup\{t \mid \Theta_{E_t^{f+g}}^\nu = 1\} \geq F + G$$

Passing to the limits as $F \rightarrow f^\nu(x)$ and $G \rightarrow g^\nu(x)$, we get

$$(f + g)^\nu(x) \geq f^\nu(x) + g^\nu(x).$$

Now we will derive the opposite inequality using Lemma 8. Indeed for almost all $x \in A$ we have:

$$-(f + g)^\nu(x) = ((-f) + (-g))^\nu(x) \geq (-f)^\nu(x) + (-g)^\nu(x) = -f^\nu(x) - g^\nu(x).$$

□

Lemma 10. *Let the function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be increasing and left-continuous. If functions f and $\phi \circ f$ are BV-similar then*

$$(\phi \circ f)^\nu(x) = \phi(f^\nu(x)) \quad (21)$$

for almost all $x \in A$.

Proof. The lemma easily follows from the equality

$$\{x \in \Omega \mid (\phi \circ f(x)) \geq \phi(t)\} = \{x \in \Omega \mid f(x) \geq t\}.$$

□

Remark 7. 1) Suppose that Hausdorff measure $H_1(\phi^{-1}(E)) = 0$ for any set E of measure 0. Then the statement that $\phi \circ f$ is BV-similar implies that the function f is BV-similar. This assertion holds definitely if (locally) $|\phi(x) - \phi(y)| \geq \text{const}|x - y|$. The last condition obviously holds if $\phi \in C^1$ and $\phi' \neq 0$.

2) In the lemma increasing of ϕ can be replaced by the assumption that the set $\phi^{-1}((t, +\infty))$ is a finite union of intervals and rays for almost all t .

Lemma 11. *If functions f , g , and fg are BV-similar then*

$$(fg)^\nu(x) = f^\nu(x)g^\nu(x). \quad (22)$$

for almost all $x \in \Gamma$.

Proof. It is enough to prove (22) only for $f, g \geq 1$. It follows from Lemma 8, Corollary 3 and the equality $f = (f^+ + 1) - (f^- + 1)$.

In this case Lemma 9, Lemma 10, and Remark 7 imply

$$(fg)^\nu = (e^{\ln(fg)})^\nu = e^{(\ln f + \ln g)^\nu} = e^{(\ln f)^\nu + (\ln g)^\nu} = e^{\ln(f^\nu) + \ln(g^\nu)} = f^\nu g^\nu.$$

□

4 Integral formula for norm of trace

Definition 2. Let us define a norm of the trace on $\partial\Omega$ of a function $f \in BV(\Omega)$ as follows:

$$\|f\|_\Gamma = \int_{\partial^*\Omega} |f^*| d\mu + \int_{\partial_\Gamma^2\Omega} (f^* - f_*) d\mu. \quad (23)$$

If $\|f\|_\Gamma < \infty$ we will say that f has the summable trace.

Lemma 12.

$$\|f\|_{\Gamma} = \|f^+\|_{\Gamma} + \|f^-\|_{\Gamma}. \quad (24)$$

Proof. We have

$$\begin{aligned} f^* - f_* &= |f^+ - f^-| = |((f^+)^+)^{\nu} - ((f^+)^-)^{\nu} - ((f^-)^+)^{-\nu} + ((f^-)^-)^{-\nu}| \\ &= |(f^+)^{\nu} - (f^+)^{-\nu}| + |(f^-)^{-\nu} - (f^-)^{-\nu}| \\ &= ((f^+)^* - (f^+)_*) + ((f^-)^* - (f^-)_*). \end{aligned}$$

□

Lemma 13. Suppose that a function $f \in BV(\Omega)$ is nonnegative and has the summable trace on $\partial\Omega$. Moreover, let a vector-function $\eta: \Gamma \rightarrow \mathbb{R}^k$, $k \geq 1$, is measurable and bounded. Then

$$\int_0^{+\infty} \int_{\Gamma \cap \partial^* E_t} \eta \, d\mu \, dt = \int_{\partial^* \Omega} f^* \eta \, d\mu + \int_{\partial_{\Gamma}^2 \Omega} (f^* - f_*) \eta \, d\mu. \quad (25)$$

Proof. Clearly, it suffices to consider only the case $k = 1$. Define

$$\begin{aligned} \{x \in \partial_{\Gamma}^1 \Omega \mid f^* > t\} &= E_t^1, \quad \{x \in \partial_{\Gamma}^2 \Omega \mid f^* > t\} = E_t^2 \\ \{x \in \partial_{\Gamma}^2 \Omega \mid f_* > t\} &= L_t^2, \quad \{x \in \partial^* \Omega \mid f^* > t\} = E_t^*. \end{aligned}$$

By (11) and Lemma 7, we have

$$\begin{aligned} \int_0^{+\infty} \int_{\Gamma \cap \partial^* E_t} \eta \, d\mu \, dt &= \int_0^{+\infty} \left(\int_{\Gamma \cap \partial_{\Gamma}^1 E_t} \eta \, d\mu - \int_{\Gamma \cap \partial_{\Gamma}^2 E_t} \eta \, d\mu \right) dt \\ &= \int_0^{+\infty} \left(\int_{E_t^1} \eta \, d\mu - \int_{L_t^2} \eta \, d\mu \right) dt \\ &= \int_0^{+\infty} \left(\int_{E_t^*} \eta \, d\mu + \int_{E_t^2} \eta \, d\mu - \int_{L_t^2} \eta \, d\mu \right) dt \\ &= \int_0^{+\infty} \left(\int_{E_t^*} \eta \, d\mu + \int_{E_t^2} \eta \, d\mu - \int_{L_t^2} \eta \, d\mu \right) dt \\ &= \int_{\partial^* \Omega} f^* \eta \, d\mu + \int_{\partial_{\Gamma}^2 \Omega} (f^* - f_*) \eta \, d\mu. \end{aligned}$$

□

Corollary 4. If a function $f \in BV(\Omega)$ is nonnegative then

$$\|f\|_{\Gamma} = \int_0^{+\infty} \mu(\Gamma \cap \partial^* E_t) \, dt. \quad (26)$$

In addition, f has the summable trace if and only if the right part of (26) is finite.

Indeed, if $\|f\|_{\Gamma} < \infty$ then we can obtain (26) substituting $\eta = 1$ in (25). Now let the right part of (26) is finite. Then it suffices to substitute $\eta = 1$ in the latter equalities of the proof of Lemma 12 and read them from right to left to prove (26).

5 Summability of traces and integral inequalities

In this and the next sections, we are going to show that in fact all the integral inequalities and other results on traces obtained in [3], [5] can be generalized to the case when the boundary of a region is a countably rectifiable set. As the integral inequalities obtained in [5] are various, we restrict ourselves with only key examples.

For a set $A \subset \bar{\Omega}$, denote by τ_A the infimum of numbers β such that the inequality $\mu(\partial^* E \cap \Gamma) \leq \beta \mu(\partial^* E \cap \Omega)$ holds for all $E \subset \Omega$ satisfying

$$\text{Vol}(A \cap E) + \mu(A \cap \partial^* E) = 0.$$

Note that τ_A goes to infinity as A vanishes. Indeed, we can set $E = \Omega \setminus A$.

The following theorem generalizes Theorem 6.5.3/1 in [5].

Theorem 1. *Let the boundary $\partial\Omega$ of a region Ω be a countably rectifiable set, D be a subset of $\bar{\Omega}$. Then for any function $f \in BV(\Omega)$ satisfying the condition $f(A \cap \Omega) = 0$, $f^*(A \cap \Gamma) = 0$ the inequality*

$$\|f\|_{\Gamma} \leq \tau_A \|f\|_{BV(\Omega)} \quad (27)$$

holds and the constant τ_A is exact.

Proof. We can assume that $\|f\|_{BV(\Omega)} < \infty$. Suppose for a while that $f \geq 0$. Note that $\text{Vol}(A \cap E_t) + \mu(A \cap \partial^* E_t) = 0$ for almost all $t > 0$. Then by Corollary (26) and the definition of τ_A we have

$$\|f\|_{\Gamma} = \int_0^{+\infty} \mu(\Gamma \cap \partial^* E_t) dt \leq \tau_A \int_0^{+\infty} P_{\Omega}(E_t) dt = \tau_A \|f\|_{BV(\Omega)}. \quad (28)$$

Let now the function f be not necessary nonnegative. By Lemma 24 we have

$$\begin{aligned} \|f\|_{\Gamma} &= \|f^+\|_{\Gamma} + \|f^-\|_{\Gamma} \\ &\leq \tau_A (\|f^+\|_{BV(\Omega)} + \|f^-\|_{BV(\Omega)}) = \tau_A \|f\|_{BV(\Omega)}. \end{aligned} \quad (29)$$

□

The next theorem generalizes Theorem 6.5.4/1 in [5].

Theorem 2. *Suppose that the boundary of a region Ω is a countably rectifiable set. Then in order for any function $f \in BV(\Omega)$ to satisfy the inequality*

$$\|f\|_{\Gamma} \leq k(\|f\|_{BV(\Omega)} + \|f\|_{L(\Omega)}) \quad (30)$$

with a constant k independent on f , it is necessary and sufficient that there exists a constant $\delta > 0$ such that the inequality

$$\mu(\partial^* E \cap \partial^* \Omega) \leq k_1 P_{\Omega}(E) \quad (31)$$

holds for every measurable set $E \subset \Omega$ with $\text{diam } E \leq \delta$, where the constant k_1 does not depend on E .

To prove the necessity of (30) it suffices to insert $f = \chi_E$ in (30). The sufficiency can be derive from Theorem 1 with the help of a partition of unity.

Theorem 4 in [3] (or, that is the same, Theorem 6.5.2(1) in [5]) can be naturally generalized to the case of regions with countably rectifiable boundary in the following form.

Theorem 3. *Let the boundary of a region Ω be a countably rectifiable set. Then the inequality*

$$\inf_c \{ \|f - c\|_\Gamma \} \leq k \|f\|_{BV(\Omega)} \quad (32)$$

is satisfied with a constant k independent on $f \in BV(\Omega)$ if and only if the inequality

$$\min \{ \mu(\Gamma \cap \partial^* E), \mu(\Gamma \cap \partial^*(\Omega \setminus E)) \} \leq k P_\Omega(E) \quad (33)$$

holds for each set $E \subset \Omega$ having the finite perimeter.

Proof. First note (cf. (11)) that

$$\mu(\Gamma \cap \partial^* E) = \mu(\partial^* \Omega \cap \partial_\Gamma^1 E) + \mu(\partial_\Gamma^2 \Omega \cap \partial^* E), \quad (34)$$

$$\mu(\Gamma \cap \partial^*(\Omega \setminus E)) = \mu(\partial^* \Omega \setminus \partial_\Gamma^1 E) + \mu(\partial_\Gamma^2 \Omega \cap \partial^* E). \quad (35)$$

Necessity. Let $E \subset \Omega$, $P_\Omega(E) < \infty$. For the characteristic function χ_E of the set E we have

$$\begin{aligned} k P_\Omega(E) &= k \|\chi_E\|_{BV(\Omega)} \\ &\geq \inf_c \left\{ \int_{\partial^* \Omega} |(\chi_E)^*(x) - c| d\mu(x) + \int_{\partial_\Gamma^2 \Omega} ((\chi_E)^*(x) - (\chi_E)_*(x)) d\mu(x) \right\} \\ &= \min_c \{ |1 - c| \mu(\partial^* \Omega \cap \partial_\Gamma^1 E) + |c| \mu(\partial^* \Omega \setminus \partial_\Gamma^1 E) + \mu(\partial_\Gamma^2 \Omega \cap \partial^* E) \} \\ &= \min \{ \mu(\partial^* \Omega \cap \partial_\Gamma^1 E), \mu(\partial^* \Omega \setminus \partial_\Gamma^1 E) \} + \mu(\partial_\Gamma^2 \Omega \cap \partial^* E). \end{aligned}$$

Jointly with (34) and (35) this proves the inequality (33).

Sufficiency. If $\|f\|_{BV(\Omega)} < \infty$ then $P(E_t) < \infty$ for almost all t . Taking into account (33)–(35), by the Fleming–Rishel formula (3) we get

$$\begin{aligned} k \|f\|_{BV(\Omega)} &= k \int_{-\infty}^{+\infty} P_\Omega(E_t) dt \\ &\geq \int_{-\infty}^{+\infty} \left(\min \{ \mu(\partial^* \Omega \cap \partial_\Gamma^1 E_t), \mu(\partial^* \Omega \setminus \partial_\Gamma^1 E_t) \} + \mu(\partial_\Gamma^2 \Omega \cap \partial^* E_t) \right) dt. \end{aligned} \quad (36)$$

Denote $t_0 = \sup \{t \mid \mu(\partial^* \Omega \cap \partial_\Gamma^1 E_t) \geq \mu(\partial^* \Omega \setminus \partial_\Gamma^1 E_t)\}$ and observe that $\mu(\partial^* \Omega \cap \partial_\Gamma^1 E_t)$ does not increase in t and $\mu(\partial^* \Omega \setminus \partial_\Gamma^1 E_t)$ does not decrease in t . Hence, by (26) we obtain

$$\begin{aligned} k\|f\|_{BV(\Omega)} &\geq \int_{t_0}^{+\infty} \mu(\Gamma \cap \partial^* E_t) dt + \int_{-\infty}^{t_0} \mu(\Gamma \cap \partial^*(\Omega \setminus E_t)) dt \\ &= \|(f - c)^+\|_\Gamma + \|(f - c)^-\|_\Gamma = \|f - c\|_\Gamma. \end{aligned}$$

So (33) holds and the theorem is proved. \square

6 Extension of a function in $BV(\Omega)$ to all the space by a constant

In this section we suppose everywhere that $P(\Omega) < \infty$ and $\partial\Omega$ is a countably rectifiable set.

Let a function f be defined in a region $\Omega \subset \mathbb{R}^n$. Denote by f_c the function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, defined by the condition $f_c(x) = f(x)$ for $x \in \Omega$ and $f_c(x) = c$ for $x \notin \Omega$, where c is a constant.

Lemma 14. *The following equality*

$$\|f_c\|_{BV(\mathbb{R}^n)} = \|f\|_{BV(\Omega)} + \|f - c\|_\Gamma \quad (37)$$

holds.

Proof. Without loss of generality we can assume $c = 0$; indeed, it is enough to consider $f - c$ instead of f . The equality (24) allows to assume that $f \geq 0$. As usual we set $E_t = \{x \in \Omega \mid f_0 > t\}$. Now by the equalities (3) and (26) we have

$$\begin{aligned} \|f_0\|_{BV(\mathbb{R}^n)} &= \int_0^{+\infty} P(\{x \in \mathbb{R}^n \mid f_0 > t\}) dt \\ &= \int_0^{+\infty} (P_\Omega(E_t) + \mu(\Gamma \cap \partial^* E_t)) dt \\ &= \|f\|_{BV(\Omega)} + \|f\|_\Gamma. \end{aligned}$$

\square

The question can arise: if it is possible to enlarge Ω by removing $\partial_\Gamma^2 \Omega$ and thus to reduce our case to one when normals in the sense of Federer exist almost everywhere on $\partial\Omega$. Sometimes it is possible. For instance, let $\Omega = D^2 \setminus \cup_{i=1}^\infty I_i$ be the disk with a sequence of intervals removed in such a way, that the sum of lengths of I_i is finite. Then every $f \in BV(\Omega)$ such that

$$\int_{\cup_{i=1}^\infty I_i} (f^* - f_*) < \infty$$

can be extended to a function $\tilde{f} \in BV(D^2)$. Unfortunately a slightly more complicated example shows that this is not necessary the case.

Example 2. Denote by $K \subset [0, 1]$ a Cantor set of positive length. Define the region Ω as follows:

$$\Omega = B_{(0,0)}(2) \setminus \{(x, y) \mid x \in [0, 1], |y| \leq (\text{dist}(x, K))^2\}. \quad (38)$$

It is not difficult to see that both of the one-sided densities equal one at all points of the set $K \times \{0\}$ and $\partial_1^2 \Omega$ is just the set of these points. Nevertheless it is impossible to enlarge Ω so that to include this set in the region.

7 Embedding theorems

The following theorem is a direct generalization of Theorem 6.5.7/1 in [5].

Theorem 4. *Suppose that $\partial\Omega$ is a countably μ -rectifiable set. Then for every function $f \in BV(\Omega)$ the inequality*

$$\left[\int_{\Omega} f^{\frac{n}{n-1}} dx \right]^{\frac{n-1}{n}} \leq nc_n^{-\frac{1}{n}} \{ \|f\|_{BV(\Omega)} + \|f\|_{\Gamma} \} \quad (39)$$

holds and the constant $nc_n^{-\frac{1}{n}}$ is exact.

Proof. By Corollary 3 and Lemma 9 we can suppose that $f \geq 0$. Just as in Theorem 7 in [3], we get

$$\left[\int_{\Omega} |f|^{\frac{n}{n-1}} dx \right]^{\frac{n-1}{n}} \leq \int_0^{+\infty} H_n(E_t)^{\frac{n-1}{n}} dt, \quad (40)$$

where as usual $E_t = \{x \in \Omega \mid f(x) > t\}$.

It follows from the isoperimetric inequality that

$$\begin{aligned} H_n(E_t)^{\frac{n-1}{n}} &\leq nc_n^{-\frac{1}{n}} P_{\mathbb{R}^n}(E_t) \\ &= nc_n^{-\frac{1}{n}} [P_{\Omega}(E_t) + \mu(\Gamma \cap \partial^*(E_t))]. \end{aligned} \quad (41)$$

Now the equations (41) and (26) imply

$$\begin{aligned} n^{-1} c_n^{\frac{1}{n}} \left[\int_{\Omega} |f|^{\frac{n}{n-1}} dx \right]^{\frac{n-1}{n}} &\leq \int_{-\infty}^{+\infty} P_{\Omega}(E_t) dt + \int_0^{+\infty} \mu(\Gamma \cap \partial^*(E_t)) dt \\ &= \|f\|_{BV(\Omega)} + \|f\|_{\Gamma}. \end{aligned}$$

□

Note that the multiplicative inequality 6.5.6 in [5] can also be generalized to our case.

8 The Gauss–Ostrogradskiy formula

Theorem 5 (The Gauss–Ostrogradskiy formula). *Let the boundary of a region Γ be a countably μ -rectifiable set. Assume that $\partial\Omega$ is equipped with a standard field ν of unit normals and the trace of a function $f \in BV(\Omega)$ is summable. Then*

$$\nabla f(\Omega) = \int_{\partial^*\Omega} f^\nu(x) \nu(x) d\mu(x) + \int_{\partial_{\mathbb{F}}^2\Omega} (f^\nu(x) - f^{-\nu}(x)) \nu(x) d\mu(x). \quad (42)$$

Proof. It suffices to prove (42) only for nonnegative functions f . Indeed, to prove the theorem in the general case it suffices to apply (42) to f^+ and f^- and then to use Corollary 3.

Obviously the right part of (42) does not depend on a choice of ν . Note that if $f^*(x) \neq f_*(x)$ then the normal to E_t in the sense of Federer at x exists for all $t \in (f_*(x), f^*(x))$ and does not depend on t . Therefore we can suppose that at each such point x the normal $-\nu(x)$ coincides with the normal to E_t in the sense of Federer for $f_*(x) < t < f^*(x)$. If we choose normals ν in such a way, the formula (42) can be rewritten in the following form:

$$\nabla f(\Omega) = \int_{\partial^*\Omega} f^*(x) \nu(x) d\mu(x) + \int_{\partial_{\mathbb{F}}^2\Omega} (f_*(x) - f^*(x)) \nu(x) d\mu(x). \quad (43)$$

Obviously, if $P(E) < \infty$ then $\nabla \chi_E(\mathbb{R}^n) = 0$. By applying (4) to the left part of (43) we obtain

$$\begin{aligned} \nabla f(\Omega) &= \int_0^\infty \nabla \chi_{E_t}(\Omega) dt \\ &= - \int_0^\infty \nabla \chi_{E_t}(\mathbb{R}^n \setminus \Omega) dt = - \int_0^\infty \nabla \chi_{E_t}(\Gamma \cap \partial^* E_t) dt. \end{aligned}$$

From the other hand, by (7) we get

$$\nabla \chi_{E_t}(\Gamma \cap \partial^* E_t) = - \int_{\Gamma \cap \partial^* E_t} \nu_{E_t}(x) d\mu(x) = - \int_{\Gamma \cap \partial^* E_t} \nu(x) d\mu(x),$$

where ν_{E_t} is the normal to E_t in the sense of Federer. Here the first equality follows from the fact that $\nu_{E_t}(x) = \nu(x)$ for almost all $x \in \Gamma \cap \partial^* E_t$, and the latter equality is true since $\mu(E_t \setminus \cup_{\tau > t} E_\tau) = 0$ for almost all $t \in \mathbb{R}$.

Therefore, applying (25) for $\eta = \nu$ we obtain

$$\begin{aligned} \nabla f(\Omega) &= - \int_0^{+\infty} \nabla \chi_{E_t}(\Gamma \cap \partial^* E_t) dt \\ &= \int_0^{+\infty} \int_{\Gamma \cap \partial^* E_t} \nu(x) d\mu(x) \\ &= \int_\Gamma f^*(x) \nu(x) d\mu(x) + \int_{\partial_{\mathbb{F}}^2\Omega} (f^*(x) - f_*(x)) \nu(x) d\mu(x). \end{aligned}$$

The theorem is proved. \square

9 Average trace of a function in $BV(\Omega)$

Let Ω be a region with the countably rectifiable boundary $\partial\Omega$. Suppose that a function $f \in BV(\Omega)$ is summable in some neighborhood of a point $x \in \Gamma$. Let us define the upper and lower average traces of the function f at x with respect to a normal ν by equations:

$$\begin{aligned}\overline{f}(x, \nu) &= \limsup_{r \rightarrow 0} 2v_n^{-1} r^{-n} \int_{B_r^\nu(x)} f(y) dy, \\ \underline{f}(x, \nu) &= \liminf_{r \rightarrow 0} 2v_n^{-1} r^{-n} \int_{B_r^\nu(x)} f(x) dy.\end{aligned}$$

If $\overline{f}(x, \nu) = \underline{f}(x, \nu)$ then their common value is called *average trace* and denoted $\tilde{f}(x, \nu)$. First we prove some properties of average traces for nonnegative functions.

Lemma 15. *Suppose that a function $f \in BV(\Omega)$ is nonnegative and locally summable. Then $\underline{f}(x, \nu) \geq f^\nu(x)$.*

Proof. (Compare with the proof of Lemma 6.6.2/1 in [5].)

Lemma 15 is obviously true if $f^\nu(x) = 0$. Suppose $0 < f^\nu(x)$. Pick $\epsilon > 0$ and choose a number t such that $0 < t < f^\nu(x)$ and $P_\Omega(E_t) < \infty$. Then $x \in \partial_\Gamma^\nu E_t$. This means that $\Theta_E^\nu(x) = 1$. Therefore there exists $r_0(x) > 0$ such that

$$1 - \epsilon < 2v_n^{-n} r^{-n} \text{Vol}(E_t \cap B_r^\nu(x)) \leq 1$$

for $0 < r < r_0(x)$. Since

$$\int_{B_r^\nu(x)} f(y) dy = \int_0^\infty \text{Vol}(E_\tau \cap B_r^\nu(x)) d\tau,$$

we obtain

$$\begin{aligned}2v_n^{-n} r^{-n} \text{Vol}(B_r^\nu(x)) \int_{B_r^\nu(x)} f(y) dy &\geq 2v_n^{-n} r^{-n} \int_0^t \text{Vol}(E_\tau \cap B_r^\nu(x)) d\tau \\ &\geq 2v_n^{-n} r^{-n} \text{Vol}(E_t \cap B_r^\nu(x)) t \geq (1 - \epsilon)t.\end{aligned}$$

Since ϵ is arbitrary we finish the proof by passing to the limit as $r \rightarrow 0$, and then by passing to the limit as $t \rightarrow f^\nu(x)$. □

Theorem 6. *If $f \in BV(\Omega)$ and $\|f\|_\Gamma < \infty$ then the average trace $\tilde{f}(x, \nu)$ of the function f exists and equals to the trace $f^\nu(x)$ almost everywhere on $\partial_\Gamma^\nu \Omega$.*

If the function is bounded, the proof is unexpectedly simple.

Lemma 16. *Let a function $f \in BV(\Omega)$ be bounded. Then the average trace $\tilde{f}(x, \nu)$ of the function f exists almost everywhere on Γ and coincides with $f^\nu(x)$.*

Proof of the lemma. Let $|f| < C$. By Lemma 20 and the equation 15 it follows that

$$f^\nu(x) = (f + C)^\nu(x) + (-C)^\nu(x) \leq \underline{(f + C)}(x, \nu) - C = \underline{f}(x, \nu).$$

Applying this inequality to $-f$, we obtain

$$(-f)^\nu(x) \leq \underline{(-f)}(x, \nu).$$

Thus, by Lemma 8 for almost all

$$f^\nu(x) \geq \overline{f}(x, \nu)$$

for almost all $x \in \Gamma$. The lemma is proved. \square

Proof of Theorem 6. As usual we may assume $f \geq 0$. Let us extend $f \in BV(\Omega)$ by zero to \mathbb{R}^n . By Lemma 14 the extended function f belongs to $BV(\mathbb{R}^n)$. Suppose that a function $f \in BV(\Omega)$ is unbounded. Let us consider the set $E = \{x \in \Omega \mid f(x) > 0\}$ and show that $\tilde{f}(x, \nu) = 0$ for almost all $x \in \Gamma \setminus \partial_\Gamma^1 E$. Recall that almost all points of $\partial\Omega$ are located on C^1 -smooth $(n-1)$ -dimensional surfaces M_i and a standard vector field ν is continuous along each M_i . For a point $x \in \Gamma \setminus \partial_\Gamma^1 E$ denote by M just the surface M_i such that $x \in M_i$. For any point $p \in M$, the surface M divides a small ball centered at p onto two open sets, U_1 and U_2 . Denote $\tilde{M} = \partial U_1 \cap \partial U_2 \subset M$. It suffices to prove that $\tilde{f}(x, \nu) = 0$ at all points $x \in \tilde{M}$ such that $\Theta_x^\nu(E) = \Theta_x^{-\nu}(E) = 0$. For the sake of definiteness, let the normals ν are directed inward of U_1 .

It is known that for U_1 and U_2 , the average trace of each function $f \in BV(U_i)$, $i = 1, 2$, equals to its trace (see [5], Theorem 6.6.2 or [3], Lemma 13). From the other hand, the trace equals zero at almost all $x \in M \setminus (\partial^*(E \cap U_1) \cap \partial^*(E \cap U_2))$. Therefore, for $i = 1, 2$

$$0 = \lim_{r \rightarrow 0} \frac{\int_{U_i \cap B_r(x)} f \, dx}{\text{Vol}(U_i \cap B_r(x))} = \lim_{r \rightarrow 0} 2v_n^{-1} r^{-n} \int_{U_i \cap B_r(x)} f \, dx. \quad (44)$$

Thus

$$\limsup_{r \rightarrow 0} 2v_n^{-1} r^{-n} \int_{B_r^\nu(x)} f \, dx \leq \limsup_{r \rightarrow 0} 2v_n^{-1} r^{-n} \int_{B_r(x)} f \, dx = 0. \quad (45)$$

Define

$$f_C(x) = \begin{cases} f(x) & \text{if } f(x) < C, \\ 0 & \text{if } f(x) \geq C, \end{cases} \quad f^C(x) = \begin{cases} 0 & \text{if } f(x) < C, \\ f(x) & \text{if } f(x) \geq C. \end{cases} \quad (46)$$

Now for almost all $x \in \Gamma \setminus \partial_\Gamma^1 E_C$ such that $0 < f^\nu(x) < C$, we have

$$\overline{f}(x, \nu) = \overline{f_C}(x, \nu) + \overline{f^C}(x, \nu) = (f_C)^\nu(x) + (f^C)^\nu(x) = f^\nu(x) + 0. \quad (47)$$

Taking into account that $\mu(\cap_{t>0} \partial_\Gamma^1 E_t) = 0$, we see that the theorem is proved. □

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